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LETTER TO THE EDITOR

Level repulsion for band 3×3 random matrices

L Molinari† and V V Sokolov‡

‡ Dipartimento di Fisica and INFN, Via Celoria 16, 20133 Milano, Italy
‡ Institute of Nuclear Physics, 630090 Novosibirsk, USSR

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Abstract. In view of the great interest in band random matrices, we investigate the level repulsion for the simplest case of 3×3 band matrices. A non-linear repulsion is found.

It is well known that for GOE random matrices, due to the rotation invariance of the ensemble, the joint distribution for the eigenvalues vanishes linearly when two eigenvalues tend to coincide. For band random matrices, which now attract a great deal of interest for the description of physically relevant statistical properties, the joint distribution of eigenvalues is not known, and is complicated to derive. For this reason we investigate the simplest case of 3×3 real symmetric matrices with matrix elements $a_{13} = a_{31} = 0$. The purpose of this short letter is to establish the character of the short-distance repulsion and the influence of the band structure. Our result is that for $|\lambda_1 - \lambda_2|$ going to zero, the joint probability for the eigenvalues behaves as

$$P(\lambda_1, \lambda_2, \lambda_3) \simeq |\lambda_1 - \lambda_2| \log(|\lambda_1 - \lambda_2|^{-1}). \tag{1}$$

To obtain the joint probability of eigenvalues one must first transform the probability measure in the matrix variables

$$\exp(-\omega \operatorname{Tr} A^2)\, \mathrm{d}a_{11}\, \mathrm{d}a_{22}\, \mathrm{d}a_{33}\, \mathrm{d}a_{12}\, \mathrm{d}a_{23} \tag{2}$$

to the new set of variables given by the eigenvalues and the angles that describe the rotation which diagonalises A, and then integrate over the angles. This is more conveniently performed by introducing the missing variable a_{13} to restore the rotation invariance of the volume element, and giving it a distribution $\delta(a_{13})$. We can then use the well known result

$$\prod_{i \leqslant j} \mathrm{d}a_{ij} = \prod_{i \leqslant j} |\lambda_i - \lambda_j| \prod_i \mathrm{d}\lambda_i\, \mathrm{d}\Omega \tag{3}$$

where in our case the invariant measure of the rotation group is given in terms of Euler angles by

$$\mathrm{d}\Omega = \sin\beta\, \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\gamma. \tag{4}$$

Then we have:

$$P(\lambda_1, \lambda_2, \lambda_3) = \prod_{i \leqslant j} |\lambda_i - \lambda_j| F(\lambda_1, \lambda_2, \lambda_3) \tag{5}$$

with $F(\lambda_1, \lambda_2, \lambda_3) = \int \delta(a_{13})\, \mathrm{d}\Omega$ depending only on the differences of eigenvalues. The matrix element a_{13} is given in terms of eigenvalues and angles by

$$\begin{aligned} &a_{13} = \sin\beta (A \sin\alpha \cos\beta + B \cos\alpha) \\ &A = (\lambda_3 - \lambda_1)\sin^2\gamma + (\lambda_3 - \lambda_2)\cos^2\gamma \qquad B = (\lambda_1 - \lambda_2)\cos\gamma \sin\gamma. \end{aligned} \tag{6}$$

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Using the Fourier representation of the delta function and integrating first over β and then over α, we get

$$F(\lambda_1, \lambda_2, \lambda_3) \simeq \int_0^{2\pi} \mathrm{d}\gamma \int_0^{\infty} \mathrm{d}k\, J_0\left(k\frac{\sqrt{A^2+B^2}-B}{2}\right) J_0\left(k\frac{\sqrt{A^2+B^2}+B}{2}\right) \tag{7}$$

where J_0 is a Bessel function. We now take, for example, the limit $|\lambda_1 - \lambda_2| \to 0$. It is easy to see that in this limit the integral over k diverges logarithmically. This means that the usual power of (linear) repulsion is weakened. For $|\lambda_1 - \lambda_2|$ small, the integral (7) is proportional to $\log|\lambda_1 - \lambda_2|$ and therefore we come to the statement given in (1).

The situation changes drastically if the delta function distribution for a_{13} is replaced with a Gaussian distribution with a finite but small dispersion σ. The only change is the appearance of a factor $\exp(-k^2\sigma^2/2)$ in the integral (7), which provides convergence in the point $\lambda_1 = \lambda_2$ to a positive non-zero limit. Therefore, for any finite σ, the joint probability (5) vanishes linearly in this limit.

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